Interactions Between Rigid-Body and Flexible-Body Motions in Maneuvering Spacecraft

Larry M. Silverberg* and Sungtae Park†
North Carolina State University, Raleigh, North Carolina

This paper describes the significant interactions between rigid-body and flexible-body motions in maneuvering spacecraft. One distinguishes between the rigid-body and flexible-body motions by introducing a tracking coordinate system that coincides with the rigid-body component of the motion, and by enforcing the motion relative to the tracking coordinate system (the elastic motion) to be orthogonal to the rigid-body motion. This leads to an infinite set of second-order weakly coupled modal differential equations describing the elastic motion. It is shown that the elastic motion is excited by the rigid-body motion through Coriolis terms, angular acceleration terms, and centrifugal terms. The Coriolis terms represent a linear time-varying gyroscopic effect, the angular acceleration terms represent a linear time-varying circulatory effect, and the centrifugal terms represent a linear time-varying stiffness effect. For unidirectional elastic motions, the Coriolis and angular acceleration terms are shown to vanish. For uniform unidirectional elastic motion, the centrifugal terms are diagonal and the modal equations become decoupled. Next, the previously indicated interactions are illustrated for spacecraft undergoing bidirectional elastic motions via the dynamics of constantly rotating free-free beams undergoing combined bending and longitudinal vibration. Finally, the dynamics of constantly rotating free-free beams undergoing bending vibration in which the stiffness operator was linearized about the static equilibrium are compared with systems in which the linearization was carried out about the dynamic equilibrium. The comparisons are made for systems rotating about axes perpendicular and parallel to the bending direction. These results indicate the sensitivity of the fundamental frequencies to linearization of the stiffness operator at high angular velocities.

I. Introductory Remarks

With the coming of larger and more flexible spacecraft tasked with missions involving rapid maneuvers, increased interest is concerned with describing the dynamic behavior of maneuvering flexible spacecraft. Moreover, the interest lies in describing the dynamic behavior in a manner that lends insight into the design process. Within this context, it does not suffice to simply compute the dynamic response of these systems, although such methods are being developed.[1,2] Instead, the interest lies in dissecting the dynamic behavior in order to reveal readily identifiable effects. Toward that end, this paper shows how the equations governing the motion of maneuvering flexible spacecraft can be decomposed into independent effects, to the extent possible. The few remaining coupling terms are then identified, and their effects on the overall motion are described.

Before proceeding with maneuvering flexible spacecraft, the next section reviews the dynamics of vibrating flexible spacecraft. The spacecraft vibratory motion will be separated into the sum of three components associated with the translations, rotations, and elastic motions, respectively. Similarly, the spacecraft forces will be separated into the sum of three components associated with the translations, rotations, and elastic behavior, respectively. For completeness, the remainder of this section demonstrates relationships between the components of motion and the linear and angular momenta as well as relationships between the components of force and the net external forces and moments acting on the spacecraft, where it is noted that each relationship is developed using closed-form expressions for the rigid-body modes of vibration along with the associated orthogonality conditions.

The third section first discusses the significance of the tracking coordinate system in describing the dynamic behavior of maneuvering flexible spacecraft.[3-5] Whereas a variety of possible tracking coordinate systems can be chosen, this section demonstrates that the equations governing the motion of maneuvering flexible spacecraft become decomposed for tracking coordinate systems chosen to coincide with the rigid-body motion of the spacecraft. Specifically, three linear independent second-order ordinary differential equations govern the three translations of the spacecraft mass center, three coupled quadratically nonlinear first-order ordinary differential equations (Euler-type equations) govern the three rotations of the spacecraft, and an infinite set of second-order weakly coupled linear modal equations govern the elastic motion of the spacecraft relative to the rigid-body motion of the tracking coordinates. This section shows that the elastic motion is excited by the rigid-body motion through three linear time-varying terms representing gyroscopic effects, stiffness effects, and circulatory effects, respectively.

Next, in Sec. IV the special case of unidirectional elastic motion is considered. For spacecraft undergoing unidirectional elastic motion, the aforementioned gyroscopic and circulatory effects are shown to vanish, and the stiffness effect is shown to lower the associated natural frequencies. As an illustration of the nature of the coupling effects, when the motion is not unidirectional, Sec. V considers the dynamics of beams rotating at a constant angular velocity undergoing combined longitudinal and bending vibration (bidirectional elastic motion).

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*Assistant Professor, Mechanical and Aerospace Engineering. Member AIAA.

†Graduate Research Assistant, Mechanical and Aerospace Engineering.

The elastic restoring forces, in the previous sections, were related to the elastic motion by a differential stiffness operator that was linearized about the static equilibrium of the spacecraft. However, as the angular velocity of the spacecraft increases, the vibration oscillates about a dynamic equilibrium rather than about a static equilibrium. To illustrate the significance of this effect, free-free beams undergoing bending motion, while rotating about a principle axis at a constant angular velocity, will expand longitudinally to a new dynamic equilibrium and then vibrate in bending about an expanded dynamic equilibrium. Then, the moments of inertia associated with the rigid-body motion will increase. Moreover, Sec. VI compares the fundamental frequencies associated with the elastic motion of free-free beams rotating about principle axes in which the stiffness operator is linearized about the static equilibrium, with the dynamic response of these beams linearized about the dynamic equilibrium instead.[6-8]

II. Dynamics of Freely Nonmaneuvering Flexible Spacecraft

Before proceeding with the development of the equations of motion for the freely maneuvering flexible spacecraft, it is desirable first to consider the equations of motion for the freely nonmaneuvering flexible spacecraft. For convenience, an inertial coordinate system is chosen with an origin that coincides with the undeformed position of the spacecraft mass center. The displacement vector of point P at time t is measured in inertial coordinates and denoted by $\underset{\sim}{u}(P,t)$. The wavy underscore denotes three-dimensional vectors. The undeformed position of point P measured relative to the origin is denoted by $\underset{\sim}{u}_C(P)=x_1\hat{i}_1+x_2\hat{i}_2+x_3\hat{i}_3$ where $\hat{i}_1,\hat{i}_2$, and $\hat{i}_3$ are unit vectors of the inertial coordinate system (see Fig. 1).

Because the spacecraft is flexible, each point P on the spacecraft is exerted upon by elastic restoring forces. The elastic restoring forces depend on the spacecraft displacement. When the displacement relative to its equilibrium is "small," only the linear part of the relationship between the elastic restoring force and the displacement is retained, and the remaining nonlinear parts are discarded. Denoting the elastic restoring force by $\underset{\sim}{f}_e(P,t)$, we obtain

$$\underset{\sim}{f}_e(P,t) = -L\underset{\sim}{u}(P,t) \tag{1}$$

where L is a self-adjoint, positive semidefinite linear differential matrix operator expressing the spacecraft stiffness. The spacecraft is also exerted upon by external forces at each point P denoted by $\underset{\sim}{f}(P,t)$. Considering Newton's laws of motion at each point P, we obtain

$$\rho(P)\ddot{\underset{\sim}{u}}(P,t) = -L\underset{\sim}{u}(P,t)+\underset{\sim}{f}(P,t) \tag{2}$$

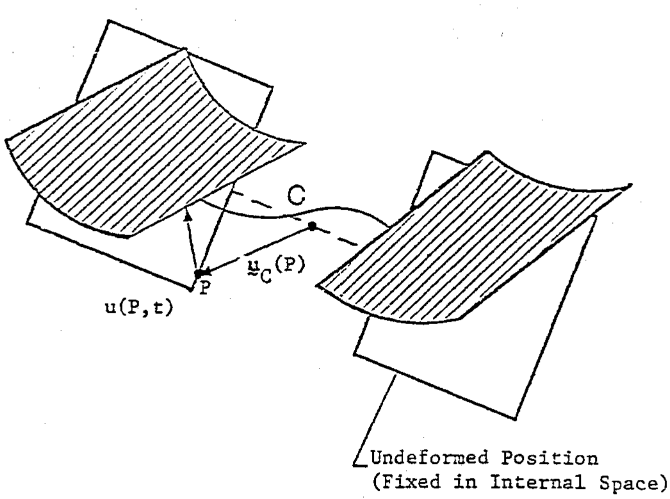

Fig. 1 Nonmaneuvering spacecraft.

where $\rho(P)$ denotes the mass density of point P, and overdots represent differentiations in time with respect to the inertial coordinate system. Because the spacecraft is freely suspended in space, the displacement is not constrained by external geometric boundary conditions. The external boundary conditions are all natural boundary conditions.

To simplify the problem, we consider the natural decompositions of the spacecraft displacement and of the external force. The displacement is expressed as a series of natural displacements, and the external force is expressed as a series of natural forces. The natural decomposition simplifies the problem because it produces a special correspondence between the natural displacements and the natural forces. To show this, we consider the eigenvalue problem associated with the stiffness operator, written

$$\lambda\rho(P)\underset{\sim}{\phi}(P) = L\underset{\sim}{\phi}(P) \tag{3}$$

The solution to Eq. (3), called the eigensolution, is composed of the nonnegative real eigenvalue λ and the associated real eigenfunction $\phi(P)$. There exists a countable number of eigensolutions, i.e., eigenvalues λ_r $(r=1,2,\ldots)$ and associated eigenfunctions $\phi_r(P)$ $(r=1,2,\ldots)$. The eigenfunctions are mutually orthogonal and can be normalized so as to satisfy the two orthonormality relations

$$\int_D\rho(P)\underset{\sim}{\phi}_r(P)\cdot\underset{\sim}{\phi}_s(P)\,\mathrm{d}D = \delta_{rs} \tag{4a}$$

$$\int_D\underset{\sim}{\phi}_r(P)\cdot L\underset{\sim}{\phi}_s(P)\,\mathrm{d}D = \lambda_r\delta_{rs}, \qquad (r,s = 1,2,\ldots) \tag{4b}$$

where δ_{rs} is the Kronecker-Delta function, and the integration is carried out over the domain D of the spacecraft.[9]

The displacement and the external force are contained in the vector space generated by the eigenfunctions. Therefore, we can express the motion and the external force as linear combinations of the eigenfunctions as

$$\underset{\sim}{u}(P,t) = \sum_{s=1}^{\infty}\underset{\sim}{u}_s(P,t), \qquad \underset{\sim}{u}_s(P,t) = \underset{\sim}{\phi}_s(P)u_s(t) \qquad (s = 1, 2, \ldots) \tag{5}$$

$$\underset{\sim}{f}(P,t) = \sum_{s=1}^{\infty}\underset{\sim}{f}_s(P,t), \qquad \underset{\sim}{f}_s(P,t) = \rho(P)\underset{\sim}{\phi}_s(P)f_s(t) \qquad (s = 1, 2, \ldots) \tag{6}$$

where $\underset{\sim}{u}_s(P,t)$ and $\underset{\sim}{f}_s(P,t)$ denote natural displacements and natural forces, respectively. The natural displacement $\underset{\sim}{u}_s(P,t)$ has spatial-dependence (shape) $\underset{\sim}{\phi}_s(P)$ identical to the eigenfunction called the natural mode of vibration, and it has time-dependence (amplitude) $u_s(t)$ called the modal displacement. The natural force $\underset{\sim}{f}_s(P,t)$ has spatial dependence $\rho(P)\phi_s(P)$ called the natural mode of force, and it has time-dependence $f_s(t)$ called the modal force. The modal displacements and the modal forces are related to the displacement and to the external force, respectively, by

$$u_r(t) = \int_D\rho(P)\underset{\sim}{\phi}_r(P)\cdot\underset{\sim}{u}(P,t)\,\mathrm{d}D$$
$$f_r(t) = \int_D\underset{\sim}{\phi}_r(P)\cdot\underset{\sim}{f}(P,t)\,\mathrm{d}D, \qquad (r = 1, 2, \ldots) \tag{7}$$

Substituting the decompositions of the motion [Eq. (5)] and of the force [Eq. (6)] into the equations of motion (2) while considering the eigenvalue problem [Eq. (3)], the equations for each natural motion are expressed as

$$\rho(P)\,\ddot{\underset{\sim}{u}}_s(P,t) = -\lambda_s\rho(P)\underset{\sim}{u}_s(P,t) + \underset{\sim}{f}_s(P,t) \qquad (s = 1, 2, \ldots) \tag{8}$$

Equations (8) are independent. Therefore, the correspondence produced by the natural decomposition is that the rth natural force can excite the rth natural displacement and no other natural displacements. Substituting the decompositions [Eqs.

(5) and (6)] into the equations of the natural displacements [Eq. (8)] yields the modal equations of motion

$$\ddot{u}_s(t) = -\omega_s^2 u_s(t) + f_s(t), \qquad (s = 1, 2, \ldots) \tag{9}$$

where the eigenvalues are related to the natural frequencies of vibration ω_r by $\lambda_r = \omega_r^2 (r = 1, 2, \ldots)$.

Of the infinite number of natural displacements, we shall, without any loss of generality, let the first three correspond to translations and the next three correspond to rotations. Together they are referred to as rigid-body displacements having natural frequencies identical to zero. The remaining infinity of natural displacements are referred to as elastic displacements having nonzero natural frequencies. To distinguish between the various types of natural displacements and natural forces, consider the following notation: The subscript T is introduced for quantities associated with translation, so that

$$\underset{\sim}{u}_{Tr}(P,t) = \underset{\sim}{u}_r(P,t), \quad \underset{\sim}{\phi}_{Tr}(P) = \underset{\sim}{\phi}_r(P), \quad u_{Tr}(t) = u_r(t) \tag{10a}$$

$$\underset{\sim}{f}_{Tr}(P,t) = \underset{\sim}{f}_r(P,t), \quad f_{Tr}(t) = f_r(t), \quad (r = 1, 2, 3) \tag{10b}$$

Then, the translational components of the displacement $\underset{\sim}{u}_T(P,t)$ and of the force $\underset{\sim}{f}_T(P,t)$ are given by

$$\underset{\sim}{u}_T(P,t) = \underset{\sim}{u}_{T1}(P,t) + \underset{\sim}{u}_{T2}(P,t) + \underset{\sim}{u}_{T3}(P,t) \tag{11a}$$

$$\underset{\sim}{f}_T(P,t) = \underset{\sim}{f}_{T1}(P,t) + \underset{\sim}{f}_{T2}(P,t) + \underset{\sim}{f}_{T3}(P,t) \tag{11b}$$

Next, the subscript R refers to quantities associated with rotations, so that

$$\underset{\sim}{u}_{Rr}(P,t) = \underset{\sim}{u}_{r+3}(P,t), \qquad \underset{\sim}{\phi}_{Rr}(P) = \underset{\sim}{\phi}_{r+3}(P)$$
$$u_{Rr}(t) = u_{r+3}(t) \tag{12a}$$

$$\underset{\sim}{f}_{Rr}(P,t) = \underset{\sim}{f}_{r+3}(P,t), \qquad f_{Rr}(t) = f_{r+3}(t)$$
$$(r = 1, 2, 3) \tag{12b}$$

Then, the rotational components of the displacement $\underset{\sim}{u}_R(P,t)$ and of the force $\underset{\sim}{f}_R(P,t)$ are given by

$$\underset{\sim}{u}_R(P,t) = \underset{\sim}{u}_{R1}(P,t) + \underset{\sim}{u}_{R2}(P,t) + \underset{\sim}{u}_{R3}(P,t) \tag{13a}$$

$$\underset{\sim}{f}_R(P,t) = \underset{\sim}{f}_{R1}(P,t) + \underset{\sim}{f}_{R2}(P,t) + \underset{\sim}{f}_{R3}(P,t) \tag{13b}$$

Finally, the subscript E refers to quantities associated with elastic displacements so that

$$\underset{\sim}{u}_{Er}(P,t) = \underset{\sim}{u}_{r+6}(P,t), \qquad \underset{\sim}{\phi}_{Er}(P) = \underset{\sim}{\phi}_{r+6}(P)$$
$$u_{Er}(t) = u_{r+6}(t) \tag{14a}$$

$$\underset{\sim}{f}_{Er}(P,t) = \underset{\sim}{f}_{r+6}(P,t), \qquad f_{Er}(t) = f_{r+6}(t)$$
$$(r = 1, 2, \ldots) \tag{14b}$$

Then, the elastic components of the displacement $\underset{\sim}{u}_E(P,t)$ and of the force $\underset{\sim}{f}_E(P,t)$ are given by

$$\underset{\sim}{u}_E(P,t) = \sum_{r=1}^{\infty} \underset{\sim}{u}_{Er}(P,t) \tag{15a}$$

$$\underset{\sim}{f}_E(P,t) = \sum_{r=1}^{\infty} \underset{\sim}{f}_{Er}(P,t) \tag{15b}$$

Now, from Eqs. (5), (6), and (10–15), the displacement of the spacecraft and the external force can be expressed as

$$\underset{\sim}{u}(P,t) = \underset{\sim}{u}_T(P,t) + \underset{\sim}{u}_R(P,t) + \underset{\sim}{u}_E(P,t) \tag{16a}$$

$$\underset{\sim}{f}(P,t) = \underset{\sim}{f}_T(P,t) + \underset{\sim}{f}_R(P,t) + \underset{\sim}{f}_E(P,t) \tag{16b}$$

Next, we consider the closed-form expressions for the rigid-body modes of vibration. The eigenfunctions associated with translations in the $\hat{i}_1$, $\hat{i}_2$, and $\hat{i}_3$ directions can be expressed in the form

$$\underset{\sim}{\phi}_{Tr}(P) = c_r \hat{i}_r, \qquad (r = 1, 2, 3) \tag{17}$$

where c_r $(r = 1, 2, 3)$ are arbitrary constants. Upon normalization, the arbitrary constants are determined, and we obtain

$$c_r = M^{-1/2}, \qquad (r = 1, 2, 3) \tag{18}$$

where $M = \int_D \rho(P)\, dD$ is the total mass of the spacecraft. The eigenfunctions associated with rotations about the $\hat{i}_1$, $\hat{i}_2$, and $\hat{i}_3$ axes can be expressed in the closed-form

$$\underset{\sim}{\phi}_{Rr}(P) = d_r \hat{i}_r \times \underset{\sim}{u}_C(P), \qquad (r = 1, 2, 3) \tag{19}$$

where d_r $(r = 1, 2, 3)$ are arbitrary constants. Upon normalization these constants are determined, with the result

$$d_r = I_{rr}^{-1/2}, \qquad (r = 1, 2, 3) \tag{20}$$

where $I_{rs} = \int_D \rho(P)[\hat{i}_r \times \underset{\sim}{u}_C(P)] \cdot [\hat{i}_s \times \underset{\sim}{u}_C(P)]\, dD$ represents the spacecraft mass moment of inertia.

Clearly, the six eigenfunctions associated with translations and rotations, as they appear, are linearly independent. Also, the three eigenfunctions associated with translations are mutually orthogonal. However, the three eigenfunctions associated with rotations are mutually orthogonal only if the inertial coordinate system coincides with the principle coordinate system. Whether the eigenfunctions associated with the rotations are mutually orthogonal or not, they and the eigenfunctions associated with the translations are orthogonal to the eigenfunctions associated with the elastic motion. Of course, if the eigenfunctions associated with rotations are not orthogonal, they can be orthogonalized, which is tantamount to locating the principle axes of the spacecraft.

To complete the development given in the previous paragraphs, it remains to show that the linear momentum of the spacecraft and the external force depend only on quantities associated with translations, and that the angular momentum of the spacecraft and the external moment depend only on quantities associated with rotations. Toward this end, the following identities can be derived using the orthonormality condition [Eq. (4)]:

$$\int_D \rho(P) \underset{\sim}{\phi}_{Rr}(P)\, dD = \underset{\sim}{0}, \qquad (r = 1, 2, 3) \tag{21}$$

$$\int_D \rho(P) \underset{\sim}{\phi}_{Er}(P)\, dD = \underset{\sim}{0}, \qquad (r = 1, 2, \ldots) \tag{22}$$

$$\int_D \rho(P) \underset{\sim}{u}_C(P) \times \underset{\sim}{\phi}_{Tr}(P)\, dD = \underset{\sim}{0}, \qquad (r = 1, 2, 3) \tag{23}$$

$$\int_D \rho(P) \underset{\sim}{u}_C(P) \times \underset{\sim}{\phi}_{Er}(P)\, dD = \underset{\sim}{0}, \qquad (r = 1, 2, \ldots) \tag{24}$$

The external force acting on the spacecraft is given by

$$\underset{\sim}{F}(t) = \int_D \underset{\sim}{f}(P,t)\, dD = \int_D \underset{\sim}{f}_T(P,t)\, dD + \int_D \underset{\sim}{f}_R(P,t)\, dD + \int_D \underset{\sim}{f}_E(P,t)\, dD \tag{25}$$

But, considering the identities in Eqs. (21) and (22), we obtain

$$\int_D \underset{\sim}{f}_R(P,t)\, dD = \underset{\sim}{0} \tag{26a}$$

$$\int_D \underset{\sim}{f}_E(P,t)\, dD = \underset{\sim}{0} \tag{26b}$$

Therefore, the rotational component and the elastic component of the force produce no external force. Then, from Eqs. (25) and (26), the external force $\underset{\sim}{F}(t)$ is given by

$$\underset{\sim}{F}(t) = \int_D \underset{\sim}{f}_T(P,t)\, dD \tag{27}$$

The external moment acting about the origin is given by

$$\underset{\sim}{M}(t) = \int_D \underset{\sim}{u}_C(P) \times \underset{\sim}{f}(P,t)\,\mathrm{d}D = \int_D \underset{\sim}{u}_C(P) \times \underset{\sim}{f}_T(P,t)\,\mathrm{d}D + \int_D \underset{\sim}{u}_C(P) \times \underset{\sim}{f}_R(P,t)\,\mathrm{d}D + \int_D \underset{\sim}{u}_C(P) \times \underset{\sim}{f}_E(P,t)\,\mathrm{d}D \quad (28)$$

Considering the identities in Eqs. (23) and (24), we obtain

$$\int_D \underset{\sim}{u}_C(P) \times \underset{\sim}{f}_T(P,t)\,\mathrm{d}D = \underset{\sim}{0} \quad (29a)$$

$$\int_D \underset{\sim}{u}_C(P) \times \underset{\sim}{f}_E(P,t)\,\mathrm{d}D = \underset{\sim}{0} \quad (29b)$$

Therefore, the translational component and the elastic component of the force produce no external moment about the mass center. Then, from Eqs. (28) and (29), the external moment $\underset{\sim}{M}(t)$ is given by

$$\underset{\sim}{M}(t) = \int_D \underset{\sim}{u}_C(P) \times \underset{\sim}{f}_R(P,t)\,\mathrm{d}D \quad (30)$$

The external linear momentum of the spacecraft is given by

$$\underset{\sim}{P}(t) = \int_D \rho(P)\dot{\underset{\sim}{u}}(P,t)\,\mathrm{d}D = \int_D \rho(P)\dot{\underset{\sim}{u}}_T(P,t)\,\mathrm{d}D + \int_D \rho(P)\dot{\underset{\sim}{u}}_R(P,t)\,\mathrm{d}D + \int_D \rho(P)\dot{\underset{\sim}{u}}_E(P,t)\,\mathrm{d}D \quad (31)$$

Considering the identities in Eqs. (21) and (22), we obtain

$$\int_D \rho(P)\dot{\underset{\sim}{u}}_R(P,t)\,\mathrm{d}D = \underset{\sim}{0} \quad (32a)$$

$$\int_D \rho(P)\dot{\underset{\sim}{u}}_E(P,t)\,\mathrm{d}D = \underset{\sim}{0} \quad (32b)$$

Therefore, the rotational component and the elastic component of the motion produce no external linear momentum. Then, from Eqs. (31) and (32), the external linear momentum is given by

$$\underset{\sim}{P}(t) = \int_D \rho(P)\dot{\underset{\sim}{u}}_T(P,t)\,\mathrm{d}t \quad (33)$$

The external angular momentum about the origin is given by

$$\underset{\sim}{H}(t) = \int_D \rho(P)\underset{\sim}{u}_C(P) \times \dot{\underset{\sim}{u}}(P,t)\,\mathrm{d}D = \int_D \rho(P)\underset{\sim}{u}_C(P) \times \dot{\underset{\sim}{u}}_T(P,t)\,\mathrm{d}D + \int_D \rho(P)\underset{\sim}{u}_C(P) \times \dot{\underset{\sim}{u}}_R(P,t)\,\mathrm{d}D + \int_D \rho(P)\underset{\sim}{u}_C(P) \times \dot{\underset{\sim}{u}}_E(P,t)\,\mathrm{d}D \quad (34)$$

Considering the identities in Eqs. (23) and (24), we obtain

$$\int_D \rho(P)\underset{\sim}{u}_C(P) \times \dot{\underset{\sim}{u}}_T(P,t)\,\mathrm{d}D = \underset{\sim}{0} \quad (35a)$$

$$\int_D \rho(P)\underset{\sim}{u}_C(P) \times \dot{\underset{\sim}{u}}_E(P,t)\,\mathrm{d}D = \underset{\sim}{0} \quad (35b)$$

Therefore, the translational component and the elastic component of the motion produce no angular momentum about the mass center. Then, from Eqs. (34) and (35), the external angular momentum about the origin is given by

$$\underset{\sim}{H}(t) = \int_D \rho(P)\underset{\sim}{u}_C(P) \times \dot{\underset{\sim}{u}}_R(P,t)\,\mathrm{d}D \quad (36)$$

III. Dynamics of Freely Maneuvering Flexible Spacecraft

In the previous section, the equations describing the dynamics of freely nonmaneuvering flexible spacecraft were derived. Using a standard "small-motion" assumption, a set of three linear partial differential equations of motion was derived and, to simplify the problem, natural decompositions of the motion and of the force were introduced. One question that arises is whether these natural decompositions of the motion and force can be considered in conjunction with the maneuvering flexible spacecraft.

It turns out that a decomposition of the motion and of the force can be considered in conjunction with maneuvering flexible spacecraft when a particular coordinate system is introduced. This coordinate system can track the translational and rotational motions of the spacecraft. Indeed, a tracking coordinate system is introduced that coincides with the rigid-body motion of the spacecraft. Note that when the spacecraft is rigid, the tracking coordinate system degenerates to a body-fixed coordinate system.

Let the origin C of the tracking coordinate system be located at the spacecraft mass center (not attached to the spacecraft). Then, the position vector of the origin C relative to the origin 0 of the inertial coordinate system is denoted by $\underset{\sim}{u}_0(t) = u_{01}(t)\hat{i}_1 + u_{02}(t)\hat{i}_2 + u_{03}(t)\hat{i}_3$, where $\hat{i}_1$, $\hat{i}_2$, and $\hat{i}_3$ are unit vectors of the inertial coordinate system. The position vector of any point P on the undeformed spacecraft relative to C is denoted by $u_C(P) = x_1\hat{b}_1 + x_2\hat{b}_2 + x_3\hat{b}_3$, where $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_3$ are unit vectors of the tracking coordinate system. Because the tracking coordinate system coincides with the rigid-body motion, the displacement of point P on the spacecraft relative to the undeformed position of point P observed in the tracking coordinate system represents an elastic displacement denoted by $\underset{\sim}{u}_E(P,t) = \underset{\sim}{u}_{Ex}(P,t)\hat{b}_1 + \underset{\sim}{u}_{Ey}(P,t)\hat{b}_2 + \underset{\sim}{u}_{Ez}(P,t)\hat{b}_3$.

Whereas the elastic motion is small, the rigid-body motion is arbitrarily large. The inertial position vector of point P relative to the origin 0 of the inertial coordinate system is given by (see Fig. 2)

$$\underset{\sim}{u}(P,t) = \underset{\sim}{u}_0(t) + \underset{\sim}{u}_C(P) + \underset{\sim}{u}_E(P,t) \quad (37)$$

The inertial displacement vector of point P is differentiated in time to obtain the inertial velocity vector of point P

$$\frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{u}(P,t) = \frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{u}_0(t) + \frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{u}_C(P) + \frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{u}_E(P,t) \quad (38)$$

in which

$$\frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{u}_C(P) = \underset{\sim}{\Omega}(t) \times \underset{\sim}{u}_C(P) \quad (39a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{u}_E(P,t) = \dot{\underset{\sim}{u}}_E(P,t) + \underset{\sim}{\Omega}(t) \times \underset{\sim}{u}_E(P,t) \quad (39b)$$

where $\mathrm{d}/\mathrm{d}t$ represents differentiation in time with respect to inertial coordinates, an overdot represents differentiation in time with respect to tracking coordinates, and $\underset{\sim}{\Omega}(t) = \Omega_1(t)\hat{b}_1 + \Omega_2(t)\hat{b}_2 + \Omega_3(t)\hat{b}_3$ denotes the angular velocity vector of the tracking coordinates. The inertial velocity vector of point P is differentiated in time to obtain the inertial acceleration vector of point P

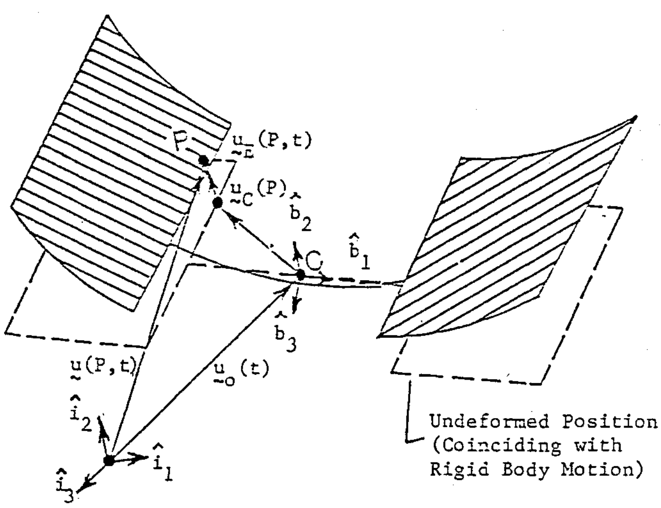

Fig. 2 Maneuvering spacecraft.

$$\frac{d^2}{dt^2}\underset{\sim}{u}(P,t)=\frac{d^2}{dt^2}\underset{\sim}{u}_0(t)+\frac{d^2}{dt^2}\underset{\sim}{u}_C(P)+\frac{d^2}{dt^2}\underset{\sim}{u}_E(P,t) \quad (40)$$

in which

$$\frac{d^2}{dt^2}\underset{\sim}{u}_C(P)=\underset{\sim}{\alpha}(t)\times\underset{\sim}{u}_C(P)+\underset{\sim}{\Omega}(t)\times[\underset{\sim}{\Omega}(t)\times\underset{\sim}{u}_C(P)] \quad (41a)$$

$$\frac{d^2}{dt^2}\underset{\sim}{u}_E(P,t)=\ddot{\underset{\sim}{u}}_E(P,t)+2\underset{\sim}{\Omega}(t)\times\dot{\underset{\sim}{u}}_E(P,t)+\underset{\sim}{\alpha}(t)\times\underset{\sim}{u}_E(P,t)+\underset{\sim}{\Omega}(t)\times[\underset{\sim}{\Omega}(t)\times\underset{\sim}{u}_E(P,t)] \quad (41b)$$

where $\underset{\sim}{\alpha}(t)=\alpha_1(t)\hat{b}_1+\alpha_2(t)\hat{b}_2+\alpha_3(t)\hat{b}_3$ denotes the angular acceleration vector of the tracking coordinates.

The spacecraft is exerted upon by elastic restoring forces $\underset{\sim}{f}_e(P,t)$ measured in tracking coordinates and, depending on the elastic displacements, in the form

$$\underset{\sim}{f}_e(P,t)=-L\underset{\sim}{u}_E(P,t) \quad (42)$$

where L is identical to the stiffness operator in the previous section. Because the stiffness operators introduced in this and the previous sections are identical, Eq. (42) represents a linearized expression about the static equilibrium of the spacecraft, which is tantamount to neglecting geometric nonlinearities and to considering only kinematic nonlinearities. Later in the paper, the stiffness operator will be linearized about the dynamic equilibrium, which is tantamount to considering geometric nonlinearities. Furthermore, the eigenfunctions obtained in the previous section will be used here with the difference that the eigenfunctions are no longer measured in inertial coordinates. Instead, the eigenfunctions are now measured in tracking coordinates. Under these assumptions, the replacement of the inertial coordinates with tracking coordinates in Eqs. (26-30), and the introduction of Eq. (38) into Eqs. (31) and (34), leads to the following properties associated with maneuvering spacecraft:

1) The elastic component of the force produces no external force on the maneuvering spacecraft.

2) The elastic component of the force produces no external moment about the maneuvering spacecraft mass center.

3) The elastic component of the motion produces no external linear momentum on the maneuvering spacecraft.

4) The elastic component of the motion produces no external angular momentum on the maneuvering spacecraft.

Considering Newton's law of motion at each point P, we obtain

$$\rho(P)\ddot{\underset{\sim}{u}}(P,t)=-L\underset{\sim}{u}_E(P,t)+\underset{\sim}{f}(P,t) \quad (43)$$

Introducing Eq. (41) into Eq. (43), we obtain the nonlinear partial differential equations describing the motion of the maneuvering spacecraft

$$\rho(P)d^2\underset{\sim}{u}_E(P,t)/dt^2=-L\underset{\sim}{u}_E(P,t)+\underset{\sim}{f}(P,t)+\underset{\sim}{n}(P,t) \quad (44)$$

in which

$$\underset{\sim}{n}(P,t)=-\rho(P)\frac{d^2}{dt^2}\underset{\sim}{u}_0(t)-\rho(P)\frac{d^2}{dt^2}\underset{\sim}{u}_C(P)-\rho(P)\times\{2\underset{\sim}{\Omega}(t)\times\dot{\underset{\sim}{u}}_E(P,t)+\underset{\sim}{\alpha}(t)\times\underset{\sim}{u}_E(P,t)+\underset{\sim}{\Omega}(t)\times[\underset{\sim}{\Omega}(t)\times\dot{\underset{\sim}{u}}_E(P,t)]\} \quad (45)$$

It is of interest to simplify the problem by expressing the motion and the force in terms of their natural decompositions and to decompose Eq. (44). By substituting Eq. (15) into Eq. (44), premultiplying by $\underset{\sim}{\phi}_{Tr}(P)$, $\underset{\sim}{\phi}_{Rr}(P)$, $(r=1,2,3)$, and by $\underset{\sim}{\phi}_{Er}(P)$, $(r=1,2,3,\ldots)$, integrating the result over the spacecraft domain and considering the orthonormality relations [Eq. (4)], we obtain the translational, rotational, and elastic modal equations of motion, respectively,

$$0=f_{Tr}(t)+n_{Tr}(t),\qquad (r=1,2,3)$$
$$0=f_{Rr}(t)+n_{Rr}(t),\qquad (r=1,2,3)$$
$$d^2u_{Er}(t)/dt^2=-\omega_r^2u_{Er}(t)+f_{Er}(t)+n_{Er}(t)\qquad (r=1,2,\ldots) \quad (46)$$

where

$$n_{Tr}(t)=\int_D\underset{\sim}{\phi}_{Tr}(P)\cdot\underset{\sim}{n}(P,t)\,dD,\qquad (r=1,2,3)$$
$$n_{Rr}(t)=\int_D\underset{\sim}{\phi}_{Rr}(P)\cdot\underset{\sim}{n}(P,t)\,dD,\qquad (r=1,2,3)$$
$$n_{Er}(t)=\int_D\underset{\sim}{\phi}_{Er}(P)\cdot\underset{\sim}{n}(P,t)\,dD,\qquad (r=1,2,\ldots) \quad (47)$$

Examination of the kinematic effects $n_{Tr}(t)$, $n_{Rr}(t)$, and $n_{Er}(t)$ associated with the translational, rotational, and elastic modal motions, respectively, remains. Substituting Eq. (45) into Eq. (47) and considering the identities in Eqs. (21-24), we obtain

$$M\frac{d^2}{dt^2}\underset{\sim}{u}_0(t)=\underset{\sim}{F}(t) \quad (48a)$$

$$I\underset{\sim}{\alpha}(t)=-\underset{\sim}{\Omega}(t)\times I\underset{\sim}{\Omega}(t)+\underset{\sim}{M}(t) \quad (48b)$$

$$\ddot{u}_{Er}(t)=-\omega_r^2u_{Er}(t)+f_{Er}(t)+n_{Er}(t),\quad (r=1,2,\ldots) \quad (48c)$$

where I denotes the spacecraft mass moment of inertia matrix with entries I_{rs} $(r,s=1,2,3)$ given earlier. The kinematic effects associated with the elastic motion can be written as

$$n_{Er}(t)=n_{Er1}(t)+n_{Er2}(t)+n_{Er3}(t),\qquad (r=1,2,\ldots) \quad (49)$$

where

$$n_{Er1}(t)=-\sum_{s=1}^{\infty}n_{Ers1}(t)\dot{u}_{Es}(t) \quad (50a)$$

$$n_{Ers1}(t)=-n_{Esr1}(t)=\int_D\rho(p)\underset{\sim}{\phi}_{Er}(P)\cdot[2\underset{\sim}{\Omega}(t)\times\underset{\sim}{\phi}_{Es}(P)]\,dD \quad (50b)$$

and

$$n_{Er2}(t)=-\sum_{s=1}^{\infty}n_{Ers2}(t)u_{Es}(t) \quad (51a)$$

$$n_{Ers2}(t)=-n_{Esr2}(t)=\int_D\rho(P)\underset{\sim}{\phi}_{Er}(P)\cdot[\underset{\sim}{\alpha}(t)\times\underset{\sim}{\phi}_{Es}(P)]\,dD \quad (51b)$$

and

$$n_{Er3}(t)=-\sum_{s=1}^{\infty}n_{Ers3}(t)u_{Es}(t)-g_{Er}(t) \quad (52a)$$

$$n_{Ers3}(t)=n_{Esr3}(t)=\int_D\{\rho(P)[\underset{\sim}{\Omega}(t)\cdot\underset{\sim}{\phi}_{Er}(P)][\underset{\sim}{\Omega}(t)\cdot\underset{\sim}{\phi}_{Es}(P)]-[\underset{\sim}{\Omega}(t)\cdot\underset{\sim}{\Omega}(t)][\underset{\sim}{\phi}_{Er}(P)\cdot\underset{\sim}{\phi}_{Es}(P)]\}\,dD \quad (52b)$$

$$g_{Er}(t)=\int_D\rho(P)\{[\underset{\sim}{\Omega}(t)\cdot\underset{\sim}{\phi}_{Er}(P)][\underset{\sim}{\Omega}(t)\cdot\underset{\sim}{u}_C(P)]-[\underset{\sim}{\Omega}(t)\cdot\underset{\sim}{\Omega}(t)][\underset{\sim}{\phi}_{Er}(P)\cdot\underset{\sim}{u}_C(P)]\}\,dD \quad (52c)$$

As can be seen from Eqs. (48a) and (48b), the translational and rotational equations are decoupled from the elastic motion. This is a direct result of selecting functions to describe the elastic motion that are orthogonal to the rigid-body motion. However, the nonlinear rigid-body rotations affect the elastic motion via the kinematic terms $n_{Er}(t)$. From Eqs. (50-52), the individual kinematic terms $n_{Er1}(t)$, $n_{Er2}(t)$, and $n_{Er3}(t)$, $(r=1,2,\ldots)$, are now recognized as Coriolis, angular acceleration, and centrifugal terms, respectively. From Eq. (50), the Coriolis terms are linear time-varying homogeneous terms multiplying the modal velocities. Moreover, the coefficients $n_{Ers1}(t)$ are skew symmetric. Therefore, the Coriolis terms are a gyroscopic effect.[9] From Eq. (51), the angular acceleration terms are linear time-varying homogeneous terms multiplying the modal displacements. Furthermore, the coefficients $n_{Ers2}(t)$

are skew symmetric. Therefore, the angular acceleration terms are a circulatory effect. From Eq. (52), the centrifugal terms have two components. One component is linear time-varying homogeneous coefficients multiplying the modal displacements, and the other component is time-varying nonhomogeneous terms. The first component is symmetric. Therefore, the first component of the centrifugal terms is a stiffness effect. The second component excites the modes of vibration. When the spacecraft rotates at a constant angular velocity about a principle axis, the second term shifts the static equilibrium of the spacecraft.

IV. Unidirectional Elastic Motion

Let us consider the special case when, at each point P on the spacecraft, the elastic motion viewed in tracking coordinates predominantly acts in one direction. Although such motions typically occur in spacecraft with relatively simple geometries, this case reveals interesting tendencies. The dominant direction is denoted by the unit vector $\underline{b}_0(P)$, implying

$$\underline{\phi}_{Er}(P) = \phi_{Er}(P)\underline{b}_0(P), \qquad (r = 1, 2, \ldots) \tag{53}$$

It follows that $\underline{\phi}_{Er}(P)\times\underline{\phi}_{Es}(P)=\underline{0}$, $(r,s = 1, 2, \ldots)$. Introducing Eqs. (53) and (54) into Eqs. (50-52), we obtain

$$n_{Er1}(t) = 0 \tag{54a}$$

$$n_{Er2}(t) = 0 \tag{54b}$$

and

$$n_{Er3}(t) = \sum_{s=1}^{\infty}\int_D \rho(P)\phi_{Er}(P)\phi_{Es}(P)\Big(\Big\{\underline{\Omega}(t)\cdot\underline{\Omega}(t) - [\underline{b}_0(P)\cdot\underline{\Omega}(t)]^2\Big\}\Big)\,\mathrm{d}D u_{Es}(t) + g_{Er}(t) \tag{54c}$$

Indeed, with the elastic motion acting unidirectionally throughout the spacecraft, the Coriolis terms and the angular acceleration terms on the elastic motion vanish. The remaining rigid-body terms on the elastic motion are due to the centrifugal motion. These terms tend to lower the natural frequency of each mode of vibration by a fraction of the magnitude of the rigid-body angular velocity. If we further assume that the unidirectional motion is uniform so that $\underline{b}_0(P) = \underline{b}_0$ is constant throughout the spacecraft, then we obtain from Eq. (54)

$$n_{Er3}(t) = \Omega^{*2}u_{Er}(t) + g_{Er}(t) \tag{55a}$$

where

$$\underline{\Omega}^{*2} = \underline{\Omega}(t)\cdot\underline{\Omega}(t) - [\underline{b}_0\cdot\underline{\Omega}(t)]^2 \tag{55b}$$

$$g_{Er}(t) = -\int_D \rho(P)\phi_{Er}(P)\underline{b}_0\cdot\Big\{\underline{\Omega}(t)\times[\underline{\Omega}(t)\times\underline{u}_c(P)]\Big\}\mathrm{d}D \tag{55c}$$

Introducing Eq. (55a) into Eq. (48c) we obtain the modal equations in the form (55c)

$$\ddot{u}_{Er}(t) = -\omega_r^{*2}u_{Er}(t) + f_{Er}(t) + g_{Er}(t), \quad (r = 1, 2, \ldots) \tag{56}$$

where the *reduced natural frequency* ω_r^* for uniform unidirectional elastic motion is given by

$$\omega_r^{*2} = \omega_r^2 - \Omega^{*2} = \omega_r^2 - \underline{\Omega}\cdot\underline{\Omega} + (\underline{b}_0\cdot\underline{\Omega})^2 \tag{57}$$

When $\underline{b}_0$ is parallel to $\underline{\Omega}(t)$, then, from Eq. (57), $\omega_r^* = \omega_r$, and when $\hat{\underline{b}}_0$ is perpendicular to $\underline{\Omega}(t)$, then $\omega_r^{*2} = \omega_r^2 - \underline{\Omega}\cdot\underline{\Omega}$. Indeed, the effect of the centrifugal motion is greatest when the angular velocity of the tracking coordinate system is perpendicular to the unidirectional elastic motion. The effect of the centrifugal motion vanishes when the directions of the angular velocity vector and the elastic motion are parallel. Furthermore, from Eq. (55a), the modal equations remain decoupled when the elastic motion is uniformly unidirectional.

V. Bidirectional Elastic Motion

As indicated in the previous section, when spacecraft vibrate unidirectionally, the Coriolis terms on the elastic motion vanish. The problem described in this section illustrates the significance of the Coriolis and centrifugal terms on the elastic motion for a system in which the elastic motion is bidirectional. We consider the combined longitudinal and bending vibration of uniform free-free beams rotating at a constant angular velocity (see Fig. 3). The elastic displacement of the beam is given by $\underline{u}_E(P,t) = u_x(x,t)\hat{\underline{b}}_1 + u_z(x,t)\hat{\underline{b}}_3$, and the stiffness operator has the form

$$L = -AE\frac{\partial^2}{\partial x^2}b_1b_1\cdot + EI_y\frac{\partial^4}{\partial x^4}b_3b_3$$

where AE and EI_y denote the uniform longitudinal stiffness and the uniform bending stiffness, respectively. This stiffness operator in this section is linearized about the static equilibrium. (The effect of linearizing the stiffness operator about dynamic equilibrium will be described later). In the absence of external forces, and assuming that the beam is rotating at a constant angular velocity $\underline{\Omega} = \Omega_y\hat{\underline{b}}_2$ about the $\hat{\underline{b}}_2$ axis, we obtain the two partial differential equations of motion:

$$m\frac{\partial^2 u_x}{\partial t^2} = AE\frac{\partial^2 u_x}{\partial x^2} + m\Omega_y^2 x - 2\Omega_y m\frac{\partial u_z}{\partial t} + m\Omega_y^2 u_x \tag{58a}$$

$$m\frac{\partial^2 u_z}{\partial t^2} = -EI_y\frac{\partial^4 u_z}{\partial x^4} + 2\Omega_y m\frac{\partial u_x}{\partial t} + m\Omega_y^2 u_z \tag{58b}$$

Next, the immediate interest is to nondimensionalize the equations of motion according to

$$\bar{u}_x = u_x/L, \qquad \bar{t} = \sqrt{EI_y/mL^4}, \qquad \bar{x} = x/L$$

$$\bar{u}_z = u_z/L, \qquad \bar{\Omega}_y = \Omega_y\sqrt{mL^4/EI_y} \tag{59}$$

Introducing Eq. (59) into Eq. (58) and replacing differentiations with respect to dimensional quantities with differentiations with respect to nondimensional quantities, we obtain

$$\frac{\partial^2\bar{u}_x}{\partial\bar{t}^2} = \frac{AL^2}{I_y}\frac{\partial^2\bar{u}_x}{\partial\bar{x}^2} + \bar{\Omega}_y^2\bar{x} - \left[2\bar{\Omega}\frac{\partial\bar{u}_z}{\partial\bar{t}}\right] + \left\{\bar{\Omega}_y^2\bar{u}_x\right\} \tag{60a}$$

$$\frac{\partial^2\bar{u}_z}{\partial\bar{t}^2} = -\frac{\partial^4\bar{u}_z}{\partial\bar{x}^4} + \left[2\bar{\Omega}_y\frac{\partial\bar{u}_x}{\partial\bar{t}}\right] + \left\{\bar{\Omega}_y^2\bar{u}_z\right\} \tag{60b}$$

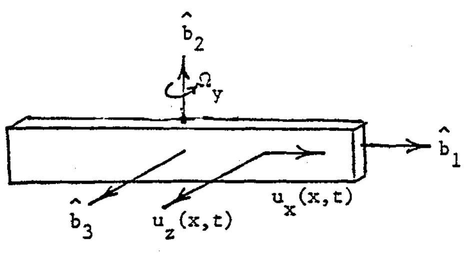

Fig. 3 Rotating beam undergoing combined bending and longitudinal vibration.

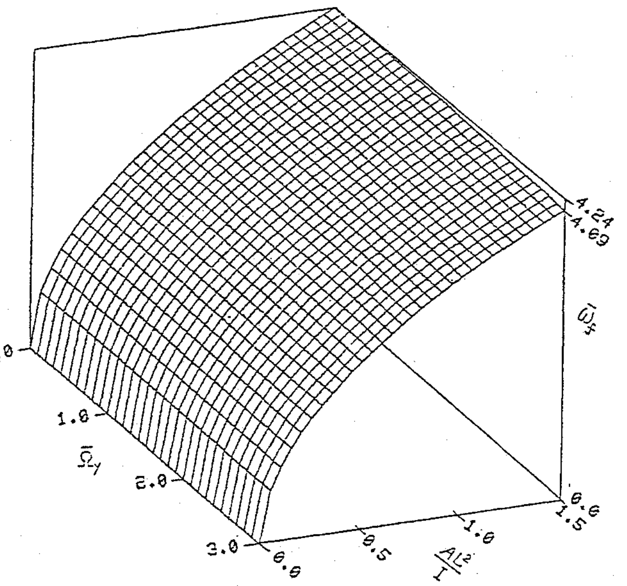

Fig. 4 Effect of Coriolis coupling on the combined bending and longitudinal vibration of a rotating free-free beam.

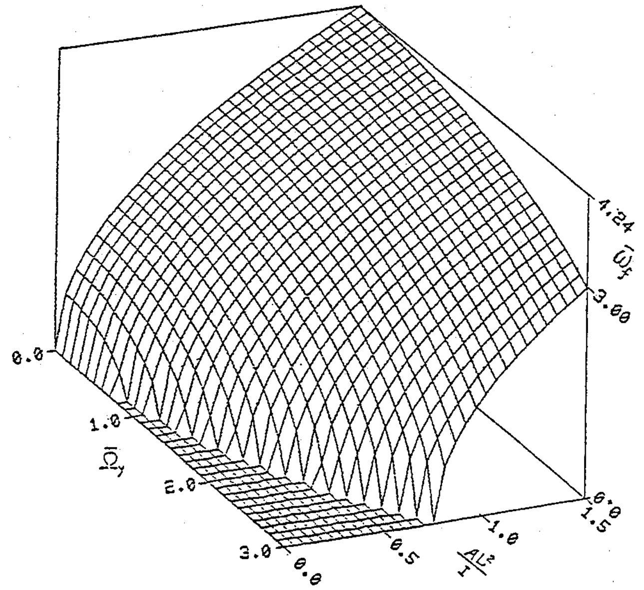

Fig. 6 Effect of Coriolis coupling and centrifugal coupling on the combined bending and longitudinal vibration of a free-free beam.

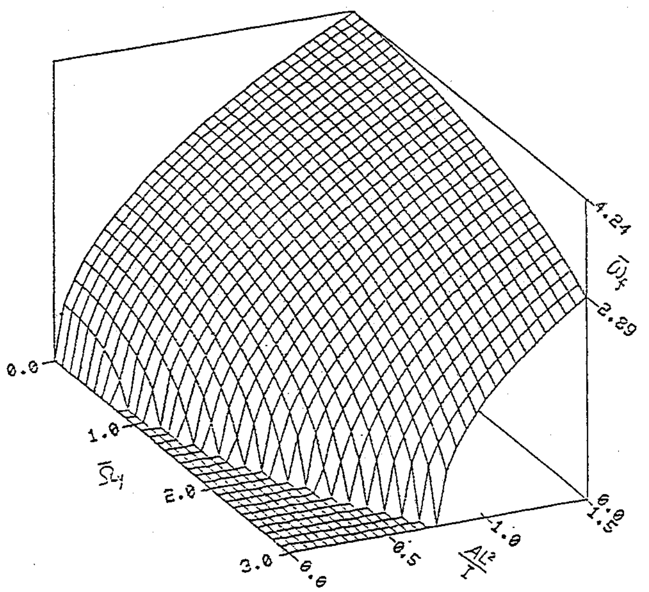

Fig. 5 Effect of centrifugal coupling on the combined bending and longitudinal vibration of a rotating free-free beam.

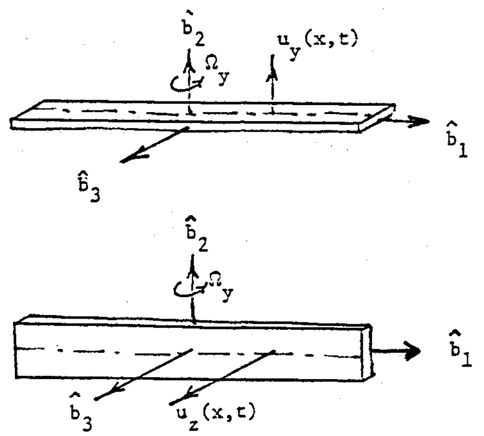

Fig. 7 Rotating beams undergoing bending vibration.

The coupling terms in the brackets [ ] represent Coriolis terms, and the coupling terms in the braces { } represent centrifugal terms. Note from Eq. (60) that the significance of these coupling terms depends on the two nondimensional parameters AL^2/I_y and $\bar{\Omega}_y$. The nondimensional parameter AL^2/I_y reflects the degree to which the elastic motion is bidirectional. Figure 4 shows the fundamental frequency of the rotating beam as a function of the two nondimensional parameters in the presence of Coriolis coupling [the centrifugal coupling in the braces is neglected in Eq. (60)]. Figure 5 shows the fundamental frequency of the rotating beam as a function of the two nondimensional parameters in the presence of centrifugal coupling [the Coriolis coupling in the brackets is neglected in Eq. (60)]. Figure 6 shows the fundamental frequency as a function of the two nondimensional parameters in the presence of both Coriolis and centrifugal coupling [no terms are neglected in Eq. (60)].

VI. Linearization of the Elastic Restoring Forces

In Sec. III the elastic restoring forces were related to the elastic motion after linearization was carried out about the static equilibrium of the spacecraft. Concurrently, in Sec. V the vibration of a beam rotating at a constant angular velocity was described in which the beam expanded longitudinally due to the centrifugal coupling term $\bar{\Omega}_y^2 \bar{x}$ in Eq. (60), so that the beam essentially vibrates in bending about an expanded dynamic equilibrium; although the stiffness operator was linearized about a static equilibrium. The question arises as to the difference between linearization about the static equilibrium and linearization about the dynamic equilibrium of maneuvering flexible spacecraft. As it turns out, the differences can be significant and affect both the rigid-body motion and the elastic motion. The rigid-body motion is affected by increased moments of inertia. The effect of linearization on the elastic motion is shown in the following two illustrations of uniform free-free beams undergoing bending vibration and rotating at a constant angular velocity (see Fig. 7). We first consider beams in which the elastic displacement is parallel to the beam's axis of rotation. The elastic displacement is given by $\underline{u}_E(P,t) = u_y(x,t)\underline{\hat{b}}_2$, and the angular velocity is given by $\underline{\Omega}(t) = \Omega_y(t)\underline{\hat{b}}_2$. The stiffness operator has the form

$$L = \left(EI_z \frac{\partial^4}{\partial x^4} - \tfrac{1}{2} m\Omega_y^2 \frac{\partial}{\partial x}\left\{\left[\left(\frac{L}{2}\right)^2 - x^2\right]\frac{\partial}{\partial x}\right\}\right)\underline{\hat{b}}_2\underline{\hat{b}}_2$$

A clear derivation of the given stiffness operator can be found in Ref. 6. In the absence of external forces, and assuming that the beam is rotating at a constant angular velocity, we obtain the partial differential equation of motion

$$m\frac{\partial^2 u_y}{\partial t^2} = -EI_z\frac{\partial^4 u_y}{\partial x^4} + \tfrac{1}{2} m\Omega_y^2\frac{\partial}{\partial x}\left\{\left[\left(\frac{L}{2}\right)^2 - x^2\right]\frac{\partial u_y}{\partial x}\right\} = 0 \quad (61)$$

Next, the equations of motion (61) are nondimensionalized according to

$$\bar{u}_y = u_y/L, \qquad \bar{t} = t\sqrt{EI_z/mL^4}, \qquad \bar{x} = x/L$$
$$\bar{\Omega}_y = \Omega_y\sqrt{mL^4/EI_z} \quad (62)$$

Introducing Eq. (62) into Eq. (61) and replacing differentiations with respect to dimensional quantities with differentiations with respect to nondimensional quantities, we obtain

$$\frac{\partial^2 \bar{u}_y}{\partial \bar{t}^2} = -\frac{\partial^4 \bar{u}_y}{\partial \bar{x}^4} + \left\{\tfrac{1}{2}\bar{\Omega}_y^2\left[\frac{\partial}{\partial \bar{x}}(\tfrac{1}{4} - \bar{x}^2)\frac{\partial \bar{u}_y}{\partial \bar{x}}\right]\right\} = 0 \quad (63)$$

The term in brackets arises when the linearization of the stiffness operator is carried out about the expanded dynamic equilibrium of the beam rather than about the undeformed static equilibrium of the beam.[10-12] Figure 8 shows the fundamental frequency of the rotating beam as a function of the nondimensionalized angular velocity. The solid line figure shows the fundamental frequency obtained after neglecting the term in brackets in Eq. (63). The dashed line figure shows the fundamental frequency obtained after including the term in brackets in Eq. (63).

Next, we consider beams in which the elastic displacement is perpendicular to the beam's axis of rotation. The elastic displacement is given by $\underline{u}_E(P,t) = u_z(x,t)\hat{\underline{b}}_3$, and the angular velocity is given by $\underline{\Omega}(t) = \Omega_y(t)\hat{\underline{b}}_2$. The stiffness operator has the form

$$L = \left\{EI_y\frac{\partial^4}{\partial x^4} - \tfrac{1}{2} m\Omega_y^2\frac{\partial}{\partial x}\left[\left(\frac{L}{2}\right)^2 - x^2\right]\frac{\partial u_z}{\partial x}\right\}\hat{\underline{b}}_3\hat{\underline{b}}_3$$

and the nonlinear accelerations in Eq. (45) are given by $\underline{n}(P,t) = m\Omega_y^2 u_z \hat{\underline{b}}_3$. In the absence of external forces and assuming that the beam is rotating at a constant angular velocity, we obtain the partial differential equation of motion

$$m\frac{\partial^2 u_z}{\partial t^2} = -EI_y\frac{\partial^4 u_z}{\partial x^4} - \tfrac{1}{2} m\Omega_y^2\frac{\partial}{\partial x}\left[\left(\frac{L}{2}\right)^2 - x^2\right]\frac{\partial u_2}{\partial x} + m\Omega_y^2 u_z \quad (64)$$

The equations of motion (64) are nondimensionalized according to

$$\bar{u}_z = u_z/L, \qquad \bar{t} = t\sqrt{EI_y/mL^4}, \qquad \bar{x} = x/L$$
$$\bar{\Omega}_y = \Omega_y\sqrt{mL^4/EI_y} \quad (65)$$

Introducing Eq. (65) into Eq. (64) and replacing differentiations with respect to dimensional quantities with differentiations with respect to nondimensional quantities, we obtain

$$\frac{\partial^2 \bar{u}_z}{\partial \bar{t}^2} = -\frac{\partial^4 \bar{u}_z}{\partial \bar{x}^4} + \left\{\tfrac{1}{2}\bar{\Omega}_y^2\frac{\partial}{\partial \bar{x}}\left[(\tfrac{1}{4} - \bar{x}^2)\frac{\partial \bar{u}_z}{\partial \bar{x}}\right]\right\}_1 + \{\bar{\Omega}_y^2\bar{u}_z\}_2 \quad (66)$$

The term in the braces $\{\ \}_1$ arises when the linearization of the stiffness operator is carried out about the expanded dynamic equilibrium rather than about the static equilibrium. The second term in the braces $\{\ \}_2$ arises due to the nonlinear acceleration in Eq. (45). Figure 9 shows the fundamental frequency of the rotating beam as a function of the nondimensional angular velocity. The solid line figure is obtained from Eq. (66) after neglecting the term in $\{\ \}_1$. The dashed line figure is obtained when all of the terms in Eq. (66) are included.

VII. Final Remarks

This paper described the significant interactions between the rigid- and flexible-body motions typically found in maneuvering spacecraft. The elastic motion was excited by the rigid-body motion through Coriolis terms, centrifugal terms, and angular acceleration terms. The Coriolis terms were shown to represent a linear time-varying gyroscopic effect. The centrifugal terms represent the sum of a linear time-varying stiffness effect, and a nonhomogeneous term shifts the spacecraft equilibrium. The angular acceleration terms represent a linear time-varying circulatory effect. For unidirectional elastic motions, the Coriolis terms and the angular acceleration terms were shown to vanish. For uniform unidirectional elastic motions, the centrifugal terms are diagonal, the modal equations are decoupled, and the associated natural frequencies decrease, depending on such factors as the angle between the bending direction and the axis of rotation.

The interactions found in spacecraft undergoing bidirectional elastic motions were illustrated via rotating beams undergoing combined bending and longitudinal vibration. The associated nondimensional fundamental frequency decreased due to the centrifugal effect more than due to the Coriolis effect by a factor of 8.

The sensitivity of the spacecraft fundamental frequency to the linearization of the stiffness operator was illustrated via

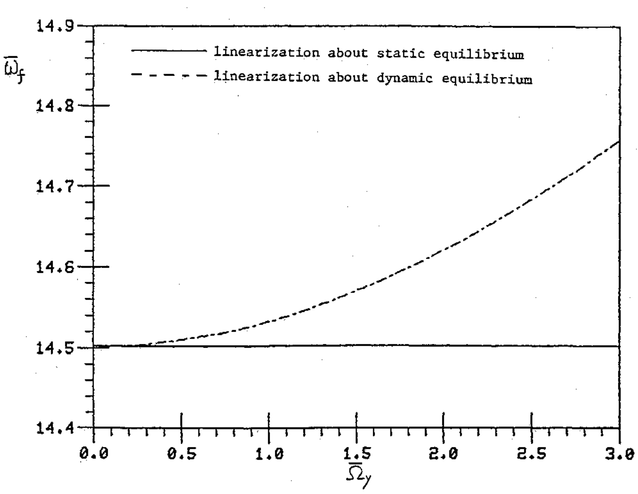

Fig. 8 Linearization of the stiffness operator for a rotating beam undergoing bending vibration parallel to axis of rotation.

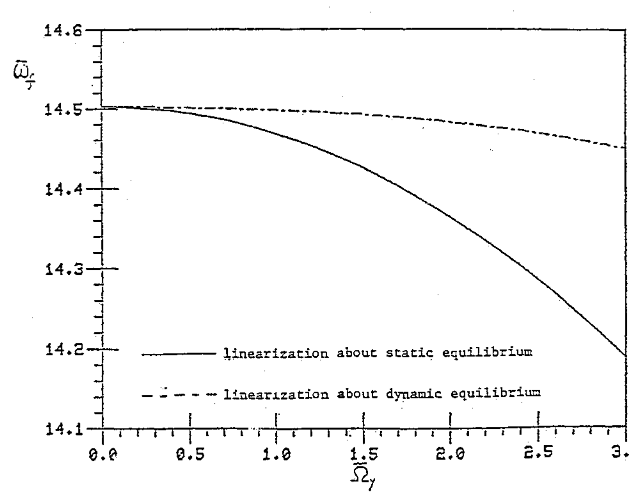

Fig. 9 Linearization of the stiffness operator for a rotating beam undergoing bending vibration perpendicular to the axis of rotation.

rotating beams undergoing bending vibration. For beams undergoing bending vibration parallel to the axis of rotation and with the linearization about the static equilibrium, the associated fundamental frequency remained constant with changes in the angular velocity of rotation. With the linearization about the static equilibrium, the fundamental frequency increased as the angular velocity of rotation increased. For beams undergoing bending vibration perpendicular to the axis of rotation, the associated fundamental frequency decreased as the angular velocity increased with the linearizations about either the static or dynamic equilibrium. With the linearization about the dynamic equilibrium, the decrease in the fundamental frequency was eight times smaller than with the linearization about the static equilibrium. Therefore, linearization about dynamic equilibrium is essential in the spacecraft maneuvering beyond angular velocities indicated in the nondimensional illustrations.

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References

[1]Simo, J. C. and Vu-Quoc, L., "On the Dynamics of Flexible Beams under Large Overall Motions—The Plane Case: Part I," *Journal of Applied Mechanics,* Paper 86-WA/APM-41, 1986.

[2]Vu-Quoc, L., and Simo, J. C., "Dynamics of Earth Orbiting Flexible Satellites with Multibody Components," *Journal of Guidance, Control, and Dynamics,* Vol. 10, Nov.-Dec. 1987, pp. 549-558.

[3]Hughes, P. C., "Dynamics of Flexible Spinning Spacecraft," XXIV Congress of the International Astronautical Federation, Baku, U.S.S.R., Oct. 1973.

[4]Canavin, J. R. and Likins, P. W., "Floating Reference Frames for Flexible Spacecraft," *Journal of Spacecraft,* Vol. 14, Dec. 1977, pp. 724-732.

[5]Laskin, R. A., Likins, P. W., and Longman, R. W., "Dynamical Equations of a Free-Free Beam Subject to Large Overall Motion," *Journal of the Astronautical Society,* Vol. 31, Oct.-Dec. 1983, pp. 507-528.

[6]Meirovitch, L., *Analytical Methods in Vibrations,* MacMillan, New York, 1967.

[7]Likins, P. W., "Mathematical Modelling of Spinning Elastic Bodies for Modal Analysis," *AIAA Journal,* Vol. 11, Sept. 1973, pp. 1251-1258.

[8]Vigneron, F. R., Comment on "Mathematical Modeling of Spinning Elastic Bodies for Modal Analysis," *AIAA Journal,* Vol. 13, Jan. 1975, p. 126; also, reply by Likins, P. W., *AIAA Journal,* Vol. 13, Jan. 1975, p. 127.

[9]Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijthoff & Noordhoff, The Netherlands, 1980.

[10]Likins, P. W., Barbera, F. J., and Baddeley, V., "Mathematical Modeling of Spinning Elastic Bodies for Modal Analysis," *AIAA Journal,* Vol. 11, Sept. 1973, pp. 1251-1258.